INTERNATIONAL JOURNAL OF NATURAL AND APPLIED SCIENCES (IJNAS), VOL. 3, NOS.1& 2 (2008); P. 86 – 88.

On delay- dependent stability test (criterion) for a retarded delay differential system

D. K. Igobi*[1], I. E. Egong[2] and C. E. Onwukwe[2]

ABSTRACT

The stability problem for a class of retarded delay differential systems is considered in this paper. The time delay (h) was assumed a variable and the asymptotic criteria was contrived. Test was formulated in the form of a scalar positive matrix functional and its application to numerical problems confirmed the suitability of this test.

INTRODUCTION

In the stability analysis of a retarded system, difficulty is experienced due to the fact that solutions of delay systems cannot be expressed explicitly. However, Lewis and Anderson (1980), Han (2001), Hale and Verduyn (1993) have published a variety of reports on the necessary and sufficient conditions of asymptotic stability of delay systems independent of the system solutions.

These conditions have been tested using different approaches. Liu and Mansour (1984), employed graphical and algebraic techniques by locating the root loci of the characteristic equation up to the stability property of the system and the distribution of the root near the delay (h). Han (2001) and Gu(2000) used the linear matrix inequalities for providing stability test. It was observed that the more sophisticated the matrix involved becomes, the more difficult it is in computing the determinant.

The present paper is an improvement on the work of Hmamed (1986) which employed a variation of the constant approach in formulating Lyapunov – Krasovskii functionals that satisfy the asymptotic stability conditions. The objective of the research was to formulate computable criteria to check the stability of a formulated functional for any given delay system and, most importantly, to generate the delay (h) variable using the linear matrix integral.

MATERIALS AND METHODS

Notations

P is a symmetric matrix (P>0). I is the identity matrix of appropriate dimension ($I \in R^{nxn}$), E^n is the n-dimensional Euclidean space and n > 0. ||.|| is the Euclidean vector norm and C [(-h, o), R^n] is the space of continuous R^n value functions on [-h, 0], $x_t \in C([-h, 0], R^n)$ is a segment of the system trajectory defined by $x_t(s) = (t+ s)$; $-h \le S \le 0$ $\|\varphi(s)\|$

Problem statement

Consider the following linear retarded delay – differential system

$$\bar{x}(t) \quad = \quad Ax(t) + Bx(t-h) \tag{1}$$

$$\bar{x}(t_o) \quad = \quad \varphi(s); \forall s \in [-h, 0] \tag{2}$$

where $x(t) \in R^n$ is the state, h > 0 is a time – delay function and $\varphi(s)$ is a continuous vector value initial function. A, B $\in R^{nxn}$ are real constant matrices. For a given initial conditions (2), system (1) admits a unique solution $x(t, t_o, \varphi(s))$ defined on t_o-h ∞)

Definition

i The solution x(t) = 0 of equation (1) is said to be stable if for any $\epsilon > 0$ there is $\delta = \delta(t_o, \epsilon) > 0$ such that if $\|\varphi(s)\| < \delta$, then $\| x(t, t_o \varphi(s)\| < \forall$ $t > t_o$

ii The solution x(t) = 0 of (i) is said to be asymptotically stable if it is stable and there exists a $\delta(t_0) > 0$ such that $x(t, t_o, \varphi(s)$ $\to 0$ as $t \to \infty$.

RESULT

We consider system (1) at the origin ($t = t_o$) such that

$$\frac{d}{dt}\left[\phi(t) - B\int_{t_0-h}^{t_o}\phi(s)ds\right] = (A+B)\phi(t) \tag{3}$$

Theorem

If there exists a scalar functional V(s, φ(s)) with continuous partial derivative calculated along the integral curve of (1) such that V(s), φ (s)) > 0 and $\overline{V}$ (s, φ(s)) < 0. Then, the equilibrium point (x(t) = 0) is Lyapunov stable.

*Corresponding author
Manuscript received by the Editor May 3, 2007; revised manuscript accepted July 20, 2007.
[1]Department of Mathematics /Statistics & Computer Science ,Cross River State University of Technology, Calabar, Nigeria
[2]Department of Mathematics /Statistics & Computer Science ,University of Calabar, Calabar, Nigeria

If the Scalar function is such that

$V(s, \varphi(s)) \to \infty$ as $\|\varphi(s)\| \to \infty$, then the equilibrium point ($x(t) = 0$) is asymptotically stable.

Remark 1

For any constant skew symmetric matrix

$M \varepsilon R^{nxn}$, $M = M^T > 0$, $X^T M X = -X^T M^T X$ and

$$K\int_0^r \eta^T(s)M\eta \quad ds \geq \left[\int_0^r \eta(s)ds\right]^T M\left[\int_0^r \eta(s)ds\right]$$ (Gu, 2000).

Remark 2

Suppose the eigenvalues of A all have negative (-) real parts (A is uniformly asymptotically stable matrix), we can choose the symmetric matrix P which is positive definite such that $A^TP + PA = -Q$ or $(A + B)^T P + P(A+B) = -Q$ and satisfy the following linear matrix inequality..

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix} > 0$$

where

$\sum_{11} = \quad (A + B)^TP + P(A + B)$

$\sum_{12} = \quad -PB + (BP)^T$

$\sum_{22} = \quad P$

Then system (1) is asymptotically stable (Han, 2001).

Proof

Considering the skew symmetric quadratic functional matrix of the form

$$V(\varphi(s)) = \varphi(s)^T P\varphi(s) \quad + \quad \int_{s-h}^{s} \varphi^T(t)P(t)\varphi(t) \quad dt \qquad (4)$$

Such that the functional $V(\varphi(s))$ from result (3) is

$V = V_1 + V_2$; where

$$V_1 = [\varphi(s) + B\int_{s-h}^{s} \varphi^T(t)dt]^T P\left[\varphi(s) + B\int_{s-h}^{s}\varphi(t)dt\right] \qquad (5)$$

$$V_2 = B\int_{s-h}^{s}\varphi^T(t)\,P\varphi(t)\,dt \qquad (6)$$

And the derivative of V along the trajectory of (3) is given as 87

$$\bar{V} = \bar{V}_1 + \bar{V}_2$$

$$\bar{V}_1 = (A+B)^T\varphi^T(s)P(\varphi(s)) + B\int_{s-h}^{s}\varphi(t)dt)$$

$$+P(A+B)\,\varphi(s)\left[(\varphi(s) + B\int_{s-h}^{s}\varphi(t)\,dt)^T\right]$$

$$= \varphi(s)\,(A+B)^TP + \quad P(A+B)\,\varphi(s) + \varphi^T(s)\,PB\,\varphi(s) \qquad (7)$$

$$V_2 = \left[\int_{s-h}^{s}\varphi(s)ds\right]^T P\left[\int_{s-h}^{s}\varphi(s)ds\right]$$

$$= \varphi^T(s)\,P\,\varphi^T(s-h) - \varphi(s)\,P\varphi(s-h)$$

$$= \varphi^T(s)\,P\,\varphi(s) - \varphi^T(s-h)\,P\varphi(s-h) \qquad (8)$$

so that

$$V(\varphi(s)) = \varphi(s)\,(A+B)^T\,P\,(A+B)\,\varphi(s) + \varphi^T(s)\,PB\varphi(s) + \varphi^T(s)\,P\varphi(s) - \varphi^T(-s-h)\,P\varphi(s-h) \qquad (9)$$

$$= -Q\varphi(s) + \varphi^T(s)\,PB\varphi(s) + \varphi^T(s)P\varphi(s) - \varphi^T(s-h)\,P\varphi(s-h) \qquad (10)$$

Equation (4) and (10) satisfies the theorem stated and that complete the proof.

Remark 3

From (3), the difference integral system $\varphi(s) - B\int_{-h}^{0}\varphi(s)ds = 0$ is asymptotically stable if there exists a $\delta > 0$ such that all roots of the characteristics equation.

$$\det\left[1 - B\int_{-h}^{o} e^{\lambda\theta}\,d\theta = 0\right]$$

Satisfy $\mathrm{Re}1[\lambda] \leq -\delta < 0$ which implies that

$$\det\left[1 - B\frac{1-e^{\lambda h}}{\lambda}\right] = 0$$

APPLICATION TO NUMERICAL PROBLEMS

Consider the time-delay system

$$\left.\begin{aligned} \bar{x}_1(t) &= -3x_1(t) - \frac{5}{2}x_2(t) + cx_2(t-h) - (i) \\ \bar{x}_2(t) &= x_1(t) + \tfrac{1}{2}x_2(t) + cx_1(t-h) - (ii) \end{aligned}\right\} \qquad (11)$$

where

$$A = \begin{bmatrix} -3 & -5/2 \\ 1 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} o & c \\ c & o \end{bmatrix}, \quad A + B = \begin{bmatrix} -3 & -5/2 + c \\ 1 + c & 1/2 \end{bmatrix}$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \qquad (Positive \quad definite)$$

and $$P_{11} = \frac{1 + 2(1+c)\left[\dfrac{17+10c}{2[15+15c-10c^2}\right]}{6}$$

$$P_{12} = \frac{17+10c}{2(15+15c-10c^2)}$$

$$P_{22} = 1 + (5-2c)\left[\frac{17+10c}{2(15+15c-10c^2)}\right]$$

The maximum time delay (h_{max}) for system (11) to assume asymptotic stability is defined as

$$h = \log(2 - \left[\sqrt{\frac{c^2-2}{c^2}}\right]^2 ; \quad 15 \leq |c| \leq 2.2$$

Therefore h_{max} for various $1.5 \leq |c| < 2.2$ values are

C	1.5	1.6	1.7	1.9	2.0	2.1	2.2
H	1.02	0.85	0.74	0.57	0.51	0.46	0.42

It is observed that for (11) to assume asymptotic stability, the |c| value must be defined as above and the delay variable decreases to 0.42. Otherwise, the system is unstable.

Example

Consider the time delay system

$$\overline{X}(t) = Ax(t) + Bx(t-h) \qquad (12)$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 1/2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1/2 \\ 1 & -0 \end{bmatrix}, \quad A + B = \begin{bmatrix} -2 & 1/2 \\ 3/2 & -2 \end{bmatrix}$$

$$\text{And } P = \begin{bmatrix} 13/4 & 2 \\ 2 & 17/4 \end{bmatrix} \text{ such that } P < 0$$

From remark 3, the maximum time delay (h_{max}) for (12) to attained asymptotic stability is defined as $h_{(max)} = ½ \log 5$. This implies that outside this value of h_{max}, system (12) will not assume asymptotic stability.

CONCLUSION

The stability problem of a linear retarded differential system has been addressed and asymptotic criteria formulated. The linear matrix integral (LMI) has been used to generate various delay intervals at which the system assumed asymptotic stability.

REFERENCES

Gu K. (2000). *An integral inequality in the stability problem of time delay systems*; Proc. 39th IEEE CDC Conf. Sydney, Australia: 2805 – 2810.

Hale J. K. and Verduyn Lunel S. M. (1993). *Introduction to Functional Differential Equation,* Springers Verlog, New York.

Han, Q. L. *(*2001). On delay-dependent stability for neutral delay-differential systems. *Int. J. Appl. Math. Comput. Sc.* 11, (4):905 – 976.

Hmamed, A.(1986). Stability conditions of delay-differential systems. *Int. J. Control*.. 43(2): 455 – 463.

Lewis, R. M. and Anderson B. D. O. (1980). *IEEE Trans. Autom. Control*, 25, 735.

Liu, Y. and Mansour, M. A.(1984).Stability test and stability conditions for delay differential system. *Int. J. Control*, , Vol. 39, No. 6: 1229 – 1242.